Mini-course on median graphs with applications to Cremona groups

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Abstract

This mini-course is dedicated to median graphs, also known as one-skeletons of CAT(0) cube complexes, and some of their applications to algebraic geometry, namely to the study of birational transformations of algebraic varieties, i.e. Cremona groups.

Contents

The notes presented here come from a mini-course given in May 2023 in Rennes for a conference organised by Serge Cantat, ERC Groups of Algebraic Transformations. For more information on the results given below, we refer the reader to [Gen] for median graphs and [LU21, GLU23] for applications to Cremona groups.

Lecture 1: A crash course on median graphs

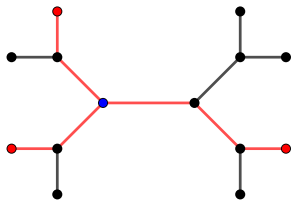
In this first lecture, we record some of the basic definitions and properties related to median graphs and groups acting on them. We focus on the results that will be relevant for the study of Cremona groups.

Definition and examples. In all the mini-course, graphs are *simplicial*: no loops nor multiple edges are allowed. Let us start by defining median graphs and discussing some examples and non-examples.

Definition 1. A connected graph X is *median* if, for all vertices $x_1, x_2, x_3 \in X$, there exists a unique vertex $m \in X$ satisfying

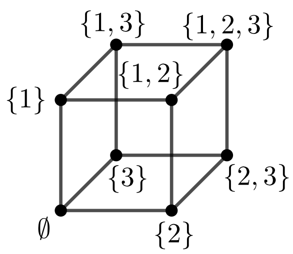
$$d(x_i, x_j) = d(x_i, m) + d(m, x_j) \text{ for all } i \neq j.$$

The vertex m is the *median point* of x_1, x_2, x_3.



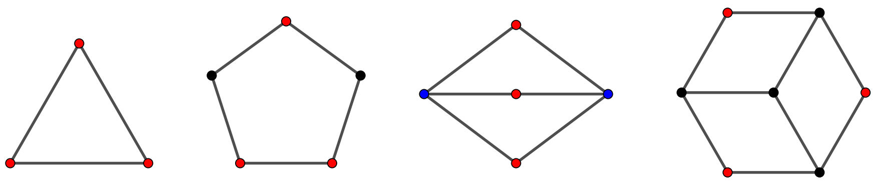
The simplest examples of median graphs are trees. Indeed, in a tree T, three vertices in generic position delimit a unique tripod, and the center of this tripod yields the median point of our three vertices.

One easily verifies that a product of median graphs is again a median graph, the median being then obtained by taking the medians of the coordinates. Thus, products of trees are also median graphs. This includes *hypercubes*.



Namely, given an arbitrary set S, the *hypercube* $\mathscr{HC}(S)$ is the graph whose vertices are the finite subsets of S (possibly empty) and whose edges connect two subsets whenever their symmetric difference has cardinality one.

The easiest examples of graphs that are not median are cycles of lengths $\neq 4$. More interesting examples include the complete bipartite graph $K_{2,3}$, which contains a triple of vertices with two distinct median points; and the *wheel* of three 4-cycles, which contains a triple of vertices with no median points.



Cubical structure. Given a median graph X, its *cube-completion* $X^\square$ is the cube complex obtained from X by filling in every subgraph isomorphic to the one-skeleton of a (hyper)cube with an actual (hyper)cube. The cube-completion of a median graph is conceptually important as it justifies that median graphs, in some sense, are obtained by gluing cubes together.

Proposition 2. *Cube-completions of median graphs are contractible.*

In fact, it turns out that cube-completions can be endowed with metrics that are, in some sense, non-positively curved. A property that can be used to characterise median graphs.

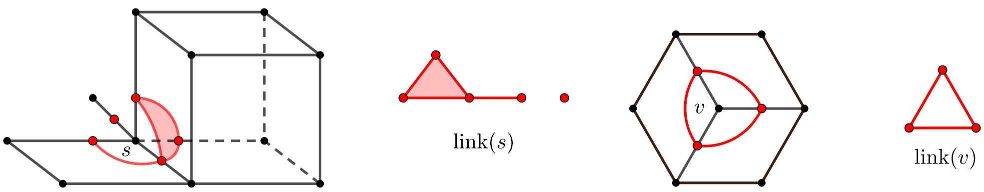
Theorem 3 ([Che00, Ger98, Rol98])**.** *The length metric obtained on the cube-completion of a median graph by extending the Euclidean metrics of the cubes is CAT(0). Conversely, the one-skeleton of a CAT(0) cube complex is a median graph.*

Thus, median graphs and CAT(0) cube complexes basically define the same objects. It is worth mentioning that, in geometric group theory, the terminology "CAT(0) cube complexes" is often used instead of "median graphs". This is a consequence of the historical origin of CAT(0) cube complexes as a convenient source of CAT(0) spaces. However, it is clear from the literature of the last decades or so that the most relevent geometry to consider is the median geometry, developped in metric graph theory during the 1960s.

For simplicity of terminology, a cube in a median graph refers to a subgraph isomorphic to the one-skeleton of an actual cube. As part of Theorem 3, cubes can be used to characterise median graphs.

Theorem 4. *A connected graph is median if and only if its cube-completion is simply connected and all its links are flag simplicial complexes.*

The *link* of a vertex v is the complex whose vertices are the edges starting from v and whose simplices are given by the edges that span a cube. The link should thought of as a small sphere around v.



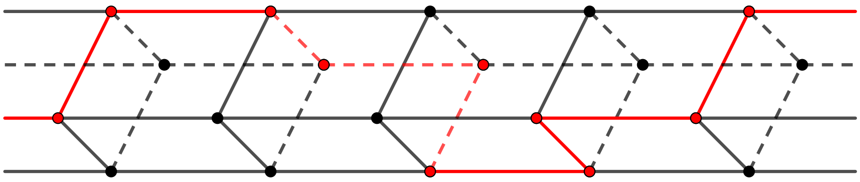
A simplicial complex is *flag* if, for every $k \geq 2$, a collection of k pairwise adjacent vertices always spans a $(k-1)$-simplex. Roughly speaking, the condition of having flag links says that every corner of a cube spans a whole cube.

Classification of isometries. In median graphs, isometries have specific behaviours, which are described by the following statement:

Theorem 5 ([Hag07, Gen])**.** *Let X be a median graph. An isometry $g \in \mathrm{Isom}(X)$ can only be*

- elliptic, *i.e. g has bounded orbits, which amounts to saying that it stabilises a cube of X;*
- loxodromic, *i.e. there exists a bi-infinite geodesic in X (referred to as an* axis*) on which g acts as a translation;*
- *or* helixodromic, *i.e. g stabilises a product $Q \times L$ of a cube Q with a bi-infinite geodesic L such that g acts on L as a translation and on Q with no fixed point.*

It is worth noticing that, most of the time, one can get rid of helixodromic isometries. Either by subdividing the median graph (so that g fixes the vertex given by the center of Q) or by replacing our isometry with a well-chosen power (since $g^{\dim(Q)!}$ fixes Q pointwise).



Consequently, Theorem 5 should be though of as a dichotomy. Let us record a few easy consequences of Theorem 5.

Corollary 6. *Let G be a group acting on a median graph X. Assume that $g \in G$ is a loxodromic isometry of X. If there exist $p, q \in \mathbb{Z} \backslash \{0\}$ such that g^p and g^q are conjugate in G, then $p = \pm q$.*

Proof. By definition, an isometry a is loxodromic if it acts as a (non-trivial) translation on some bi-infinite geodesic. Let $\tau(a)$ denote the corresponding translation length. Notice that $\tau(a^n) = |n| \cdot \tau(a)$ for every $n \in \mathbb{Z}$. Moreover, τ is conjugacy-invariant. Consequently, if $g \in G$ is loxodromic and if g^p, g^q are conjugate in G, then

$$|p| \cdot \tau(g) = \tau\left(g^p\right) = \tau\left(g^q\right) = |q| \cdot \tau(g),$$

hence $p = \pm q$ since $\tau(g) \neq 0$. □

For instance, Corollary 6 prevents groups acting properly on median graphs to contain a Baumslag-Solitar group $\mathrm{BS}(p,q) = \langle a, t \mid ta^pt^{-1} = a^q \rangle$ for some integers $p \neq q$.

Corollary 7. *Let G be a group acting on a median graph X. For every loxodromic isometry $g \in G$, there exists some $N \geq 0$ such that g does not admit a kth root in G for every $k \geq N$.*

Proof. Let $h \in G$ be an element satisfying $h^k = g$ for some $k \geq 1$. Then

$$\tau(g) = \tau\left(h^k\right) = k \cdot \tau(h) \geq k,$$

where $\tau(\cdot)$ denotes the translation length as previously defined. □

For instance, Corollary 7 prevents groups acting properly on median graphs to contain groups such as $\mathbb{Q}$ or $\mathbb{Z}[1/p]$.

Fixed-point theorems. The natural step following the classification of isometries provided by Theorem 5 is the classification of group actions (beyong the cyclic case). Let us mention a couple of statements in the direction of actions with bounded orbits. First of all:

Theorem 8 ([Ger98])**.** *Let G be a group acting on a median graph X. If G has bounded orbits, then it stabilises a cube in X.*

A particular case of this statement will be proved during the third lecture. As an application:

Corollary 9. *Let G be a group acting properly and cocompactly on a median graph X. Then G contains only finitely many conjugacy classes of finite subgroups.*

Proof. As a consequence of Theorem 8, every finite subgroup of G stabilises a cube. Up to conjugacy, we can assume that such a cube belong to some finite fundamental domain. Conversely, every cube-stabiliser is finite. Therefore, the stabilisers of the cubes from a fundamental domain provide a finite collection of finite subgroups containing a conjugate of every finite subgroup of G. □

Next, a natural question to ask is whether a group all of whose elements have bounded orbits must have a bounded orbit. In full generality, the answer is negative. For instance, every countable group that is not finitely generated admits an action on a tree with unbounded orbits. But, even if the group is finitely generated, the answer may be negative. For instance, there exist finitely generated torsion groups (such as the Grigorchuk group $\mathfrak{G}$, the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{G}$, or some Burnside groups) that act on median graphs with unbounded orbits. However, such median graphs are typically infinite hypercubes. Our next statement shows that this essentially the only obstruction:

Theorem 10 ([GLU23])**.** *Let G be a finitely generated group acting on a median graph X with no infinite cube. If the action is* purely elliptic *(i.e. if every $g \in G$ has bounded orbits), then G stabilises a cube in X.*

A particular case of this statement will be proved during the third lecture. As an immediate application of the theorem:

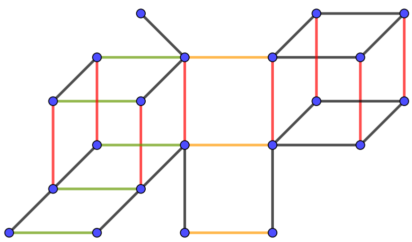
Corollary 11. *Given a group acting properly on a median graph with no infinite cube, every finitely generated torsion group is finite.*

Proof. As a consequence of Theorem 10, a finitely generated torsion group must stabilise a cube. But, by assumption, cube-stabilisers are finite. ☐

Hyperplanes. A fundamental tool in the study of median graphs is the notion of *hyperplanes*.

Definition 12. In a median graph, a *hyperplane* is an equivalence class of edges with respect to the reflexive-transitive closure of the relation that identifies two opposite edges in a 4-cycle.

One can think of hyperplanes as parallelism classes of edges.



Two hyperplanes are *transverse* if they contain two parallel edges in some 4-cycle. For instance, on the left graph, the red hyperplane is transverse to the green and yellow hyperplanes.

The idea to keep in mind is that hyperplanes encode the geometry of median graphes. This is motivated, in particular, by the following statement:

Theorem 13 ([Sag95])**.** *Let X be a median graph.*

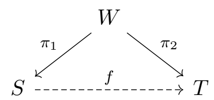
- *The graph $X \backslash\backslash J$ obtained from X by removing the (interiors of the) edges in J has exactly two connected components, referred to as* halfspaces*, which are convex.*
- *The* carrier *of J, i.e. the graph induced by J, is convex.*
- *A path in X is geodesic if and only if it crosses each hyperplane at most once.*
- *The distance between two vertices coincides with the number of hyperplanes separating them.*

Lecture 2: Median graphs for Cremona groups in dimension two

The aim of the second lecture is to provide an example of a median graph, called blow-up graph, for groups of birational transformations of surfaces, in order to illustrate the first lecture. We will also list a series of results we can obtain from this action.

Recall on Cremona groups In this mini-course, a variety X over a field k is an integral and separated scheme of finite type over k. In this lecture we consider surfaces, i.e. varieties of dimension 2. A birational map $f\colon X \dashrightarrow Y$ is an isomorphism between two open dense subsets $U \subset X$ and $V \subset Y$. For a variety X we denote by $\mathrm{Bir}(X)$ the group of birational transformations from X to itself. If X is a rational variety of dimension n, the group $\mathrm{Bir}(X)$ is isomorphic to the Cremona group in n variables $\mathrm{Bir}(\mathbb{P}^n_k)$.

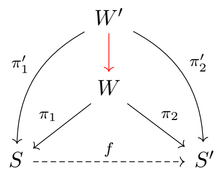
Theorem 14 ([Sta19, Tag 0C5Q])**.** *Let S and T be projective regular surfaces over a field k and let $f\colon S \dashrightarrow T$ be a birational transformation. Then there exists a projective surface W over k and two morphisms $\pi_1\colon W \to S$, $\pi_2\colon W \to T$ such that $f = \pi_2\pi_1^{-1}$:*



and we can factorize $\pi_1\colon W \to S^n \to \cdots \to S^1 \to S^0 = S$ and $\pi_2\colon W \to T^m \to \cdots \to T^1 \to T^0 = T$, where each morphism is a blow-up in a closed point.

Remark 15. If S is a regular surface, then Bl_p is again regular ([**?**, SGA6, Expose VII, Proposition 1.8]). So in particular, if S and T are projective regular surfaces, then all the surfaces $S^1, \ldots, S^n$, $W, T^1, \ldots, T^m$ appearing in the factorization given by Theorem 14 are projective and regular.

Remark 16. W can be chosen minimal and we call it *minimal resolution* of f, in the sense that for any other projective regular surface W' satisfying Theorem 14:



there exists a surjective morphism $\pi\colon W' \to W$.

This remark allows us to define the set of base points of f.

Definition 17. Let $f\colon S \dashrightarrow T$ be a birational transformation between two projective regular surfaces over a field k. The *base points* of f is the set of closed points that are blown-up by π_1 in the minimal resolution of f. We denote this set by $\mathcal{B}(f)$.

Example 18. Consider $\sigma : (x, y) \mapstochar\dashrightarrow (\frac{1}{x}, \frac{1}{y})$ the standard quadratic involution (see Figure 1).

Construction of the blow-up graph and first properties

Definition 19. The *blow-up graph* associated to a projective regular surface S over k, denoted by $\mathcal{C}(S)$ is the graph obtained as follows.

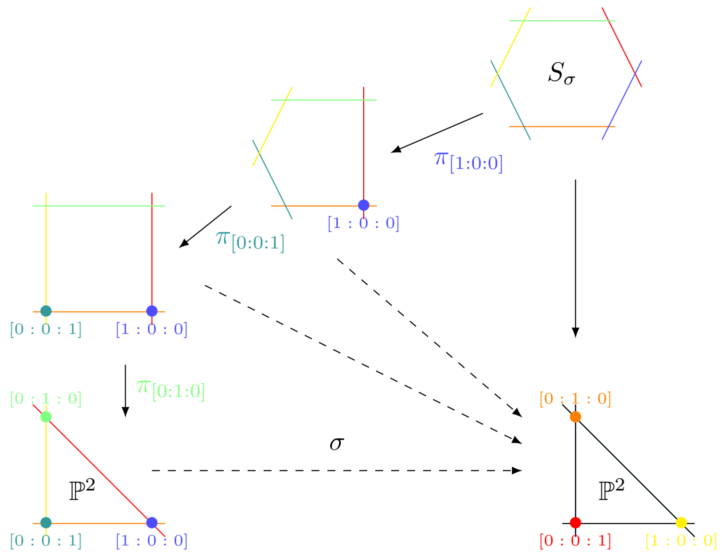


Figure 1: Resolution of σ.

- The vertices are equivalent class of marked surfaces $[(T, \varphi)]$
  - T projective regular surface over k,
  - $\varphi : T \dashrightarrow S$ birational map,
  - $(T, \varphi) \sim (T', \varphi')$ iff $\varphi'^{-1}\varphi : T \xrightarrow{\sim} T'$ is an isomorphism.
- There is an edge between two vertices $[(W, \varphi)]$ and $[(T, \psi)]$ if $\psi^{-1}\varphi$ is a blow-up or the inverse of a blow-up of a closed point.

Example 20. $[(\mathbb{P}^2, \mathrm{id})] = \{(\mathbb{P}^2, \alpha) \mid \alpha \in \mathrm{PGL}(3, k)\}$.

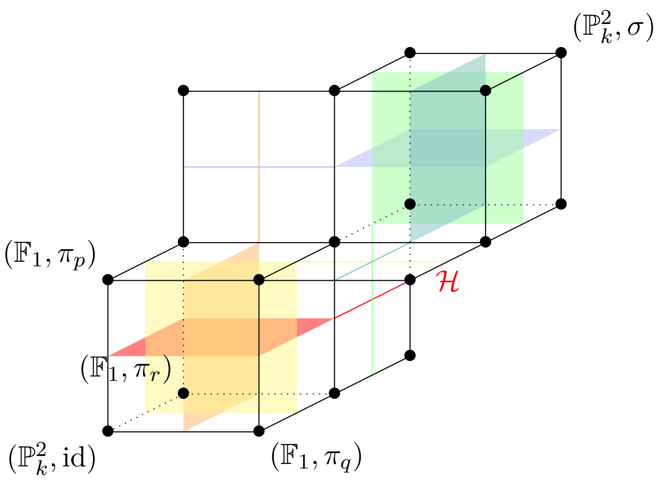


Figure 2: A subcomplex of $\mathcal{C}^\square(\mathbb{P}^2_k)$: the convex hull of the set of vertices $\{(\mathbb{P}^2_k, \mathrm{id}), (\mathbb{P}^2_k, \sigma)\}$, where σ is the standard quadratic involution.

Remark 21. The cube completion of the blow-up graph $\mathcal{C}^\square(S)$, called the *blow-up complex*, is obtained by filling the span graph of the vertices $[(T_1, \varphi_1)], \ldots, [(T_{2^n}, \varphi_{2^n})]$ with an n-cube, if there exists $1 \leq r \leq 2^n$ such that for any $1 \leq j \leq 2^n$:

- $p_1, \dots, p_n \in T_r$ distinct closed points,
- $\varphi_r^{-1}\varphi_j : T_j \to T_r$ is the blow-up of $E \subset \{p_1, \dots, p_n\}$.

Example 22. Figure 2 illustrate the convex hull of $\{(\mathbb{P}^2_k, \mathrm{id}), (\mathbb{P}^2_k, \sigma)\}$ in $\mathcal{C}^{\square}(\mathbb{P}^2_k)$.

Properties 23. *The blow-up complex $\mathcal{C}^{\square}(S)$ is:*

- *oriented: from $[(S, \varphi)]$ to $[(S', \varphi')]$ if $\varphi^{-1}\varphi'$ is the blow-up of a closed point,*
- *not locally compact,*
- *infinite dimensional (even worse: contains infinite cube).*

Theorem 24. *The blow-up graph is a median graph.*

Proof. By Theorem 4, we need to show the following.

Simple-connectness. First, it is connected by Zariski theorem 14. Then take a loop and push it in the 1-skeleton without backtrack. We call it γ. Take $w = [(T, \psi)]$ a vertex that dominates all the vertices, i.e. for any vertex $v_i = [(S_i, \varphi_i)]$ of γ, $\varphi_i^{-1}\psi$ is a morphism. Such a vertex exists by Zariski's theorem 14. The loop γ is homotopic to w. Indeed, we do an induction on the complexity of the vertices of γ defined as follows: for any vertex v_i of γ, $c(v_i)$ is the number of closed points blown-up by $\varphi_i^{-1}\psi$. Consider a vertex v_j of maximal complexity. Then v_{j-1} and v_{j+1} are obtained from v_j by blowing-up two distinct closed points p_{j-1} and p_{j+1}. Replace in γ the vertex v_j by the vertex v_j' obtained by blowing up p_{j-1} and p_{j+1}. Then w still dominate v_j' and either the complexity has decreased either the number of vertices realising the maximal complexity has decreased. We conclude by repeating the process.

Links of vertices are flag. Let v be a vertex of $\mathcal{C}(S)$. The vertices of $\mathrm{link}(v)$ can be identified with the set of edges $\{v, w\}$ in $\mathcal{C}(S)$. Let $\{v, w_1\}, \dots, \{v, w_n\}$ be a set of pairwise adjacent vertices in $\mathrm{link}(v)$, i.e., the vertices v, w_i, w_j belong to a square in $\mathcal{C}(S)$. We want to show that $\{v, w_1\}, \dots, \{v, w_n\}$ span a simplex in $\mathrm{link}(v)$, i.e., that the vertices $v, w_1, \dots, w_n$ belong to a cube.
Let T be the surface corresponding to v and for $1 \leq i \leq n$ let S_i be the surface corresponding to the vertex w_i. We denote by $m \in \{0, \dots, n\}$ the index such that, up to relabeling the vertices, for $m < j \leq n$, the edges $\{v, w_j\}$ correspond to the contraction of some curves E_j on T and for $1 \leq i \leq m$, the edges $\{v, w_i\}$ correspond to the blow-up of different closed points p_i on T. Since the $\{v, w_j\}$ are pairwise adjacent in $\mathrm{link}(v)$, it follows that the E_j are pairwise disjoint. Using also that the $\{v, w_i\}$ are pairwise adjacent in $\mathrm{link}(v)$, none of the points p_i for $1 \leq i \leq m$ can lie on one of the curves E_j for $m < j \leq n$. This implies that there exists a cube containing $\{v, w_1, \dots, w_n\}$. □

An edge is given by the blow-up of a closed point p belonging to a marked surface (W, φ). We denote this edge by (W, φ, p) and by $[(W, \varphi, p)]$ its equivalence class as well as the hyperplane dual to it. We can characterize as follows hyperplanes.

Lemma 25. *Two edges (W, φ, p) and (W', φ', q) correspond to the same hyperplane if and only if $\varphi'^{-1}\varphi$ induces a local isomorphism between a neighborhood of p and a neighborhood of q and $\varphi'^{-1}\varphi(p) = q$.*

For instance in the example of Figure 2, the hyperplane $\mathcal{H}$ corresponds to the hyperplane $[(\mathbb{P}^2_k, \mathrm{id}, p)] = [(\mathbb{F}_1, \pi_q, \pi_q^{-1}(p))]$, where $\mathbb{F}_1$ denotes the first Hirzebruch surface.

Definition 26. Let S be a surface over a field k. We say that a subgroup $G < \mathrm{Bir}(S)$ is *regularisable* if there exists a projective regular surface T over k and a birational transformation $\Psi : T \dashrightarrow S$ such that $\Psi^{-1}G\Psi \subset \mathrm{Aut}(T)$. An element $f \in \mathrm{Bir}(S)$ is *regularisable* if the subgroup generated by f is regularisable.

Let $f \in \mathrm{Bir}(S)$ and $[(T,\varphi)]$ be a vertex, the action is given by left composition:

$$f \bullet [(T,\varphi)] = [(T, f\varphi)].$$

Remark 27. By Theorem 5, we have the dichotomy: either an isometry is elliptic and it is equivalent to the property of being regularisable, or it is loxodromic. Moreover a subgroup $G < \mathrm{Bir}(S)$ is regularisable if and only if it fixes a vertex.

Proposition 28. *The distance between the vertices* $[(T,\varphi)]$ *and* $[(T',\varphi')]$ *is equal to:*

$$\mathrm{d}\left((T,\varphi),(T',\varphi')\right) = |\mathcal{B}(\varphi'^{-1}\varphi)| + |\mathcal{B}(\varphi^{-1}\varphi')|.$$

Remark 29. Theorem 13 gives indeed more: all the possible orders in the minimal resolution (Theorem 14) give all the possible geodesic paths. See for instance Figure 2 for all the possible geodesics joining the vertices $[(\mathbb{P}^2_k, \mathrm{id})]$ and $[(\mathbb{P}^2_k, \sigma)]$.

The *translation length* of an isometry f of the blow-up graph is the number

$$\ell(f) = \min\{d(f(x),x) \mid x \in V(\mathcal{C}(S))\},$$

where $V(\mathcal{C}(S))$ is the set of vertices of $\mathcal{C}(S)$. The set of vertices realizing the translation length is called the *minimizing set* of f and it is denoted by

$$\mathrm{Min}(f) = \{x \in V(\mathcal{C}(S)) \mid \mathrm{d}(x, f(x)) = \ell(f)\}.$$

Proposition 28 and Theorem 5 allow us to extend the following results to any field.

Proposition 30. *Let S be a projective regular surface over a field k and let $f \in$* $\mathrm{Bir}(S)$*.*

- $\ell(f) = \lim\limits_{n\to\infty} \frac{|\mathcal{B}(f^n)|}{n} + \frac{|\mathcal{B}(f^{-n})|}{n}$ *(*dynamical number of base-points *introduced by J. Blanc and J. Déserti [BD15]).*
- *if (S',φ) belongs to* $\mathrm{Min}(f)$ *then $\varphi^{-1}f\varphi$ is algebraically stable. In particular, every birational transformation is conjugate to an algebraically stable transformation ([DF01]).*

Applications to regularisation's results

Proposition 31. *Let S be a projective regular surface over a field k. A subgroup $G \subset$* $\mathrm{Bir}(S)$ *is regularisable if and only if there exists a constant K such that $|\mathcal{B}(f)| \leq K$ for all $f \in G$.*

The number of base points of an element $g \in \mathrm{Bir}(\mathbb{P}^2_k)$ is bounded by a constant depending only on the degree of g. Hence, as corollary, we obtain the following well-known result, which is usually shown using results on the Zariski-topology on $\mathrm{Bir}(\mathbb{P}^2_k)$, Weil's regularisation theorem, and Sumihiro's results on equivariant completions:

Corollary 32. *Let k be a field and $G \subset$* $\mathrm{Bir}(\mathbb{P}^2_k)$ *be a group such that* $\deg(g) \leq K$ *for all $g \in G$, then G is regularisable.*

A group G has the *property FW* if every action of G on a median graph has a fixed point. The class of groups with property FW is rather rich and contains in particular all groups with Kazhdan's property (T), such as $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$.
An element g in a group G is called *divisible*, if for every integer $n \geq 0$ there exists an element $f \in G$ such that $f^n = g$. An element $g \in G$ is called *distorted*, if $\lim_{n\to\infty} \frac{|g^n|_S}{n} = 0$ for some finitely generated subgroup $\Gamma \subset G$ containing g, where $|g^n|_S$ denotes the word length of g^n in Γ with respect to some finite set S of generators of Γ.

Corollary 33. *Let S be a surface over a field k. Subgroups of* $\mathrm{Bir}(S)$ *with the FW property and divisible and distorted elements of* $\mathrm{Bir}(S)$ *are regularisable.*

As a corollary we also obtain that if a subgroup of $\mathrm{Bir}(S)$ is regularisable over the algebraic closure, then it is already regularisable over k, when k is a perfect field:

Theorem 34. *Let S be a geometrically irreducible surface over a perfect field k and let $G \subset \mathrm{Bir}(S)$ be a subgroup. Consider the algebraic closure $\bar{k}$ of k and let $S_{\bar{k}} = S \times_k \bar{k}$. Then the following are equivalent:*

1. *There exists a projective regular surface $T_{\bar{k}}$ over $\bar{k}$ and a $\bar{k}$-birational transformation $\varphi \colon T_{\bar{k}} \dashrightarrow S_{\bar{k}}$ such that $\varphi^{-1}G\varphi \subset \mathrm{Aut}(T_{\bar{k}})$.*
2. *There exists a projective regular surface T' over k and a k-birational transformation $\varphi \colon T' \dashrightarrow S$ such that $\varphi^{-1}G\varphi \subset \mathrm{Aut}(T')$.*

Proof. After taking a completion and a resolution of singularities we may assume that S is projective and regular. Since k is perfect and since S is geometrically irreducible, the scheme $T'_{\bar{k}}$ is a regular projective surface over $\bar{k}$, and $\mathrm{Aut}(T') \subset \mathrm{Aut}(T'_{\bar{k}})$, in particular, if $\varphi G \varphi^{-1} \subset \mathrm{Aut}(T')$, then $\varphi G \varphi^{-1} \subset \mathrm{Aut}(T'_{\bar{k}})$. So 2 implies 1. Now assume that 1 is satisfied. We denote by $|\mathcal{B}_k(f)|$ and $|\mathcal{B}_{\bar{k}}(f)|$ the number of base points of f considered as a birational transformation defined over k and $\bar{k}$ respectively. By Proposition 31, $\mathcal{B}_{\bar{k}}(f) \leq K$ for all $f \in G$ for some global constant K. But for any $f \in G$, we have $\mathcal{B}_k(f) \leq \mathcal{B}_{\bar{k}}(f)$. Hence, $\mathcal{B}_k(f) \leq K$ for all $f \in G$, and by Proposition 31 we obtain 2. □

We focus now in the following open question.

Question 35. Let S be a surface over a field k and G be a finitely generated subgroup of $\mathrm{Bir}(S)$ such that for any $g \in G$, g is regularisable. Is G regularisable?

In term of the action of the Cremona group on the blow-up graph, Question 35 can be reformulated as follows:

Question 36. Consider a finitely generated subgroup G of the Cremona group such that each of its elements is elliptic for the action on the blow-up graph. *Does it implies that G fixes a vertex of the blow-up graph?*

We saw in the first lecture that finitely generated groups acting purely elliptically on a median graph without infinite cube have a fixed point. The problem here is that the blow-up complex has infinite cubes... But modifying it a bit we can obtain partial results to Question 35.

Theorem 37 ([GLU23])**.** *Let k be a finite field, S a surface over k and $G < \mathrm{Bir}(S)$ a finitely generated subgroup such that every element in G is regularisable, then G is regularisable.*

Proof. After taking a completion and a resolution of singularities we may assume that S is projective and regular. Let F be a finite field extension of k such that all the base-points of a symmetric finite system of generators of G are rational. We consider a subgraph of the blow-up graph, called *the rational blow-up graph* defined as the convex hull of the vertices $[(T, \varphi)]$ such that the base-points of φ and φ^{-1} are rational. Being a convex subgraph of a median graph it is also median. And by definition, the edges are given by blowing up rational closed points, hence as we are over a finite field it is locally compact and does not contain any infinite cube. G acts on this complex by construction, hence by Theorem 8, G is regularisable. □

An element $f \in \mathrm{Bir}(\mathbb{P}^2)$ is *algebraic*, if $\deg(f^n)$ is uniformly bounded for all $n \in \mathbb{Z}$. A subgroup $G \subset \mathrm{Bir}(\mathbb{P}^2)$ is *bounded* if the degree of all elements in G is uniformly bounded. Clearly, a bounded subgroup consists of algebraic elements. However, the converse is not true. For instance, consider the subgroup defined by affine coordinates (x, y) of $\mathbb{P}^2$ by

$$G = \{(x + p(y), y) \mid p \in k(y)\}.$$

Theorem 38 ([LPU])**.** *Let k be a field and $\Gamma < \mathrm{Bir}(\mathbb{P}^2)$ a finitely generated subgroup such that every element in Γ is algebraic. Then Γ is bounded.*

As a consequence of [Can11] who proved that a finitely generated subgroup $\Gamma < \mathrm{Bir}(\mathbb{P}^2)$ consisting of algebraic elements is either bounded or preserves a rational fibration, our contribution consists in proving Theorem 38 for finitely generated groups that preserve a rational fibration.

Generalisation in higher dimension One could be tempted to try to construct a median graph whose vertices are marked projective varieties of dimension 3. However, this is not possible. The group of monomial birational transformations of $\mathbb{P}^3_k$ is isomorphic to $\mathrm{GL}_3(\mathbb{Z})$ and has the property FW. However, it is known that the group of monomial transformations on $\mathbb{P}^3_k$ is not conjugate to any subgroup of the automorphism group of a projective variety. This can be seen, for instance, by considering the degree sequence of the monomial transformation $(x, y, z) \dashrightarrow (yx^{-1}, zx^{-1}, x)$, which does not satisfy any linear recurrence and is therefore not conjugate to an automorphism of a regular projective threefold (see [HP07] for details).

Lecture 3: Median graphs, the return

The goal of this lecture is to prove the following statement, which is a particular case of Theorem 10 above.

Theorem 39. *Let G be a finitely generated group acting on a locally finite median graph X of finite cubical dimension. If $G \curvearrowright X$ is purely elliptic, then G stabilises a cube.*

Warm-up: finite orbits. We start with an easy case, namely we begin by proving that a group acting on a median graph (possibly locally infinite) with a finite orbit must stabilise a cube. The proof of this statement will be a rather straightforward of two fundamental properties of median graphs.

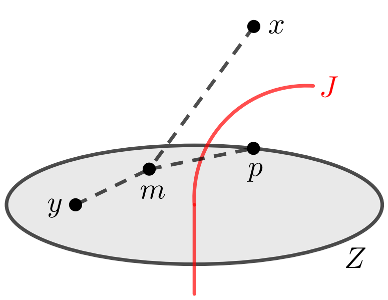
The first property claims that there exist canonical projections onto convex subgraphs. From this, we will deduce a characterisation of convex hulls in median graphs.

Proposition 40. *Let X be a median graph and $Y \subset X$ a convex subgraph. For every $x \in X$, there exists a vertex $p \in Y$ such that, for every $y \in Y$, some geodesic connecting x to y passes through p. The vertex p is the unique vertex of Y minimising the distance to x.*

Proof. Fix a vertex $p \in Y$ minimising the distance to x. If $y' \in Y$ is another such vertex, let m denote the median point of p, y, y'. Because m belongs to a geodesic between y and y', the convexity of Y assures that $m \in Y$. And, because m belongs to a geodesic between p and y (resp. y'), we have $d(p,m) \leq d(p,y)$ (resp. $\leq d(p,y')$), with equality if and only if $m = y$ (resp. $m = y'$). Since y and y' minimise the distance to x, we must have $y = m = y'$. Thus, there exists a unique vertex of Y minimising the distance to x. Let p denote this vertex. Given an arbitrary vertex $y \in Y$, the same argument shows that p is the median point of x, p, y, which amounts to saying that p belongs to a geodesic from x to y. □

Corollary 41. *Let X be a median graph and $Y \subset X$ a set of vertices. The convex hull of Y coincides with the intersection of all the halfspaces containing Y.*

Proof. Because halfspaces are convex, it is clear that the convex hull of Y is contained in the intersection of all the halfspaces containing Y. Conversely, given a vertex $x \in X$ that does not belong to the convex hull Z of Y, we claim that there exists a halfspace containing Y but not x. Let p denote the projection of x onto Z. Because x does not belong to Z, $p \neq x$, so there exists at least one hyperplane J separating x from p.



In order to conclude, it suffices to show that J separates x from Z (and a fortiori from Y). If not, there exists a vertex $y \in Z$ separated from p by J. Let m denote the median point of x, y, p. Because m belongs to a geodesic, the convexity of Z assures that m belongs to Z. Because m belongs to a geodesic between x and p, we have $d(x,m) \leq d(x,p)$, which implies that $m = p$. But the convexity of the halfspace delimited by J and containing x, y implies that m is separated from p by J, hence $m \neq p$. □

The second property we need is the *Helly property* for convex subgraphs:

Proposition 42 (Helly property)**.** *Let X be a median graph and $Y_1, \ldots, Y_n \subset X$ a finite collection of convex subgraphs. If $Y_i \cap Y_j \neq \emptyset$ for all $1 \leq i,j \leq n$, then $\bigcap_{1 \leq i \leq n} Y_i \neq \emptyset$.*

Proof. First assume that $n = 3$, and fix three vertices $x \in Y_1 \cap Y_2$, $y \in Y_2 \cap Y_3$, and $z \in Y_3 \cap Y_1$. Let $m \in Y_3$ denote the median point of $\{x, y, z\}$. Because m belongs to a geodesic between x and y, we deduce from the convexity of Y_2 that $m \in Y_2$; similarly, it follows from the convexity of Y_1 and Y_3 that $m \in Y_1$ and $m \in Y_3$. Thus, we have proved that m belongs to $Y_1 \cap Y_2 \cap Y_3$. A fortiori, $Y_1 \cap Y_2 \cap Y_3 \neq \emptyset$.

The general case $n \geq 3$ follows by induction. $\square$

We are now ready to prove the announced fixed-point property:

Corollary 43. *Let G be a group acting on a median graph X. If G has a finite orbit, then it has to stabilise a cube.*

Proof. First of all, let us record the following consequence of Corollary 41:

Claim 44. *The convex hull in X of a finite set of vertices is finite.*

Let S be a finite set of vertices and let Z denote its convex hull. According to Corollary 41, the hyperplanes crossing Z are the hyperplanes separating at least two vertices of S. Because S is finite, and since only finitely many hyperplanes separate two given vertices, it follows that Z is a median graph with only finitely many hyperplanes. But such a median graph is necessarily finite. Indeed, if $\mathcal{D}$ denotes the set of all the halfspaces of Z, the map

$$\left\{ \begin{array}{ccc} Z & \to & 2^{\mathcal{D}} \\ z & \mapsto & \{\text{halfspaces containing } z\} \end{array} \right.$$

is injective since any two distinct vertices are separated by at least one hyperplane. This concludes the proof of Claim 44.

Now, fix a finite G-orbit F. We knwo that its convex hull Z is G-invariant and finite. For every hyperplane J of Z, say that J is *balanced* if its two halfspaces have the same size. Otherwise, it is *unbalanced*, and we denote by J^+ its larger halfspace. Clearly, two such halfspaces cannot be disjoint, so it follows from the Helly property that

$$Q := \bigcap_{J \text{ hyperplane of } Z} J^+$$

is non-empty. By construction, the hyperplanes crossing Q are the balanced hyperplanes of Z. But two balanced hyperplanes, say J and K, must be transverse: otherwise, they would delimit two disjoint halfspaces, both of size $|Z|/2$, which would imply that there are no vertices between J and K. Therefore, the hyperplanes of Q are pairwise transverse. The only such median graphs are hypercubes. Thus, we have found our G-invariant cube in X. $\square$

Pairwise transverse hyperplanes. It is clear that an n-cube in a median graph yields a collection of n pairwise transverse hyperplanes. Let us show that, conversely, every collection of pairwise transverse hyperplanes meet in an n-cube.

Proposition 45. *Let X be a median graph. If $J_1, \ldots, J_n$ are pairwise transverse hyperplanes, then there exists a cube whose hyperplanes are precisely $J_1, \ldots, J_n$.*

Proof. Because $J_1, \ldots, J_n$ are pairwise transverse, the carriers $N(J_1), \ldots, N(J_n)$ pairwise intersect. Because carriers pairwise intersect and are convex, the Helly property implies that there exists a vertex $x \in X$ which belongs to all the $N(J_i)$. By construction, for every $1 \leq i \leq n$, there exists a neighbour x_i of x such that J_i is the only hyperplane separating x and x_i. Let Q denote denote the convex hull of $x, x_1, \ldots, x_n$. As a consequence of Corollary 41, the hyperplanes of Q are exactly $J_1, \ldots, J_n$. Because having all its hyperplanes transverse characterises cubes, it follows that Q is the cube we are looking for. $\square$

Corollary 46. *The cubical dimension of a median graph coincides with the maximal number of pairwise transverse hyperplanes.*

Proof. If a median graph X contains an n-cube, the its hyperplanes yield n pairwise transverse hyperplanes. Conversely, if a median graph contains n pairwise transverse, then it follows from Proposition 45 that the median graph contains an n-cube. $\square$

Recall that the Ramsey theorem states that, for every $n \geq 1$, there exists some $R(n) \geq 1$ such that the following holds. Given a complete graph with $R(n)$ vertices and whose edges are coloured using two colours, one can find a subgraph of size n all of whose edges are coloured by the same colour. Combining the Ramsey theorem and Corollary 46, we deduce:

Proposition 47. *Let X be a median graph of finite cubical dimension d. Every collection of $\geq R(d+1)$ hyperplanes contains a subcollection of pairwise non-transverse hyperplanes.*

Proof. Consider the complete graph whose vertices are the hyperplanes of our collection. Colour an edge connected two transverse hyperplanes red and blue otherwise. We know from the Ramsey theorem that there exists a monochromatic subgraph of size $d+1$. But as a consequence of Corollary 46, there is no collection of $d+1$ pairwise transverse hyperplane, so we get a collection of $d+1$ pairwise non-transverse hyperplanes. $\square$

Proof of the theorem. So far, the assumption that our group is finitely generated has not been used. It plays a role in our next statement.

Lemma 48. *Let G be a group acting on a median graph. If G is finitely generated, then there exists a convex subgraph $Y \subset X$ on which G acts with finitely many orbits of hyperplanes.*

Proof. Fix an arbitrary vertex $o \in X$ and let Y denote the convex hull of the orbit $G \cdot o$. As a consequence of Corollary 41, the hyperplanes crossing Y coincide with the hyperplanes separating at least two vertices in $G \cdot o$. Consequently, given a hyperplane J crossing Y, there exist $g, h \in G$ such that J separates go and ho. Given a finite generating set $S \subset G$, we can write $g^{-1}h$ as a product $s_1 \cdots s_n$ of elements in S. Necessarily, J separates two consecutive vertices along

$$g \cdot o, gs_1 \cdot o, \ldots, gs_1 \cdots s_{n-1} \cdot o, gs_1 \cdots s_{n-1}s_n \cdot o = h \cdot o,$$

say $gs_1 \cdots s_k \cdot o$ and $gs_1 \cdots s_k s_{k+1} \cdot o$. This implies that $(gs_1 \cdots s_k)^{-1}J$ separates o and $s_{k+1} \cdot o$.

Thus, we have proved that every hyperplane crossing Y has a G-translate in

$$\bigcup_{s \in S} \{\text{hyperplanes separating } o \text{ and } s \cdot o\}.$$

Because S is finite, and since any two vertices are separated only by finitely many hyperplanes, this set has to be finite, concluding the proof. $\square$

We are now ready to prove our theorem.

Proof of Theorem 39. According to Lemma 48, we can assume that there are only finitely many G-orbits of hyperplanes up to replacing X with a G-invariant convex subgraph. We distinguish two cases.

First, assume that G has bounded orbits. Then, because X is locally finite, it has a finite orbit. We conclude from Corollary 43 that G stabilises a cube.

Next, assume that G has unbounded orbits. So, given an arbitrary vertex $o \in X$, there exists some $g \in G$ such that $d(o, go)$ is larger than $R(2N+1)$ where N denotes the number of orbits of hyperplanes. So we get a collection $\mathcal{H}$ of $\geq R(2N+1)$ hyperplanes separating o and go. According to Proposition 47, $\mathcal{H}$ contains a subcollection containing at least $2N+1$ pairwise non-transverse hyperplanes. Among these hyperplanes, we can find three of them in the same G-orbit, say A, B, C. Let A^+ (resp. B^+, C^+) denote the halfspace delimited by A (resp. B, C) containing go and A^- (resp. B^-, C^-) its complement. If there exists some $g \in G$ such that $gA^+ = B^+$, then it follows from $gA^+ \subsetneq A^+$ that g has unbounded orbits, which impossible. Similarly, there cannot be some $h \in G$ such that $hB^+ = C^+$. Therefore, there exist $g, h \in G$ such that $gA^+ = B^-$ and $hB^+ = C^-$. But then

$$hgA^+ = hB^- = (hB^+)^c = (C^-)^c = C^+.$$

From $hgA^+ \subsetneq A^+$, it follows that hg has unbounded orbits, a contradiction. □

Lecture 4: Median graphs for Cremona groups in higher dimensions

Recall that, given a graph Γ, the *right-angled Artin group* $A(\Gamma)$ is defined by the presentation

$$\langle u \in V(\Gamma) \mid [u,v]=1 \ (\{u,v\} \in E(\Gamma)) \rangle$$

where $V(\Gamma)$ and $E(\Gamma)$ denote the vertex- and edge-sets of Γ. Right-angled Artin groups interpolate between free groups (when Γ has no edge) and free abelian groups (when Γ is complete). Given an arbitrary group G acting on a median graph X with no *inversion* (i.e. with no isometry stabilising a hyperplane and switching its two halfspaces), our goal now is to construct a natural morphism from G to some right-angled Artin group.

We fix once for all an orientatin of the hyperplanes of X (i.e. we orient the edges of X so that two opposite sides in a 4-cycle have parallel orientations). Let Γ denote the graph whose vertices are the G-orbits of hyperplanes in X and whose edges connect two orbits whenever they contain transverse hyperplanes. Notice that an oriented path α in X is naturally labelled by the word written over $V(\Gamma) \sqcup V(\Gamma)^{-1}$ given by the oriented hyperplanes successively crossed by α. Fixing a vertex $o \in X$, we claim that

$$\Theta : \left\{ \begin{array}{ccc} G & \to & A(\Gamma) \\ g & \mapsto & \text{label of a path from } o \text{ to } g \cdot o \end{array} \right.$$

defines a morphism.

The fact that Θ is well-defined, i.e. the fact that the element of $A(\Gamma)$ represented by the label of path only depends on its endpoints follows from Lemma 49 below. Indeed, adding or removing a backtrack to a path amouts to adding or removing a subword uu^{-1} or $u^{-1}u$ (where $u \in V(\Gamma)$) to its label. And flipping a 4-cycle amounts to replacing a subword uv (resp. $u^{-1}v$, uv^{-1}, $u^{-1}v^{-1}$) with vu (resp. vu^{-1}, $v^{-1}u$, $v^{-1}u^{-1}$) where $\{u,v\} \in E(\Gamma)$. Thus, Θ is well-defined.

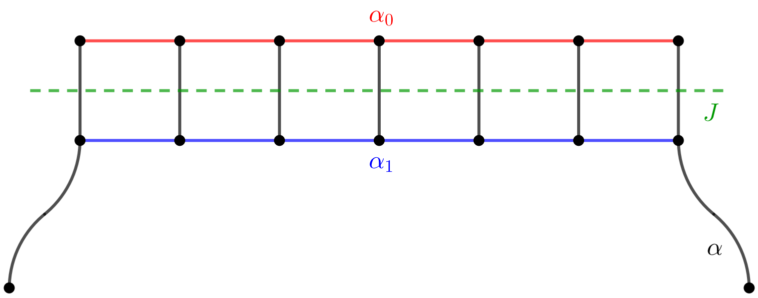
In order to justify that Θ is a morphism, let $g,h \in G$ be two elements and fix two oriented paths $[o,go]$ and $[o,ho]$. Then $[o,go] \cup g[o,ho]$ yields an oriented path from o to gho, and the label of $g[o,ho]$ coincides with the label of $[o,ho]$ because two edges in the same G-orbit have the same label. Hence

$$\begin{aligned} \Theta(gh) &= \text{label of } [o,go] \cup g[o,ho] = (\text{label of } [o,go]) \cdot (\text{label of } g[o,ho]) \\ &= (\text{label of } [o,go]) \cdot (\text{label of } [o,ho]) = \Theta(g)\Theta(h) \end{aligned}$$

Lemma 49. *Let X be a median graph and α, β two paths with the same endpoints. Then α can be transformed into β by adding or removing backtracks and by* flipping 4-cycles*.*

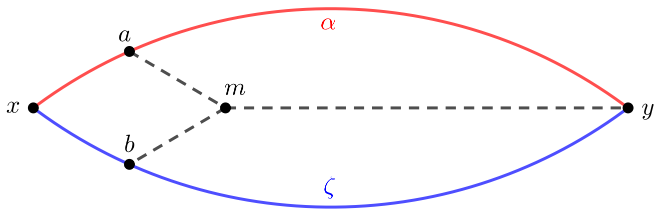
Here, given a path $\gamma := (x_0, \ldots, x_n)$, we refer to a path obtained from γ by *flipping a 4-cycle* as a path by replacing some vertex x_i with the fourth vertex of 4-cycle spanned by x_{i-1}, x_i, x_{i+1}.

Proof of Lemma 49. First, we claim that α can be made geodesic by removing backtracks and flipping 4-cycles. Indeed, according to Theorem 13, if α is not geodesic then it crosses some hyperplane J twice. Let α_0 denote a subsegment of α lying between two edges of J. Without loss of generality, we assume that α_0 does not cross a hyperplane twice. Consequently, α_0 is a geodesic. By convexity of halfspaces and carriers, α_0 has a mirror path α_1 on the other side of J.



Clearly, we can replace α_0 in α with α_1 by flipping 4-cycles and removing one backtrack. The process decreases the length of α. After finitely many iterations, we obtain a geodesic. This proves our first claim.

Now, assuming that α is a geodesic, we fix an arbitrary geodesic ζ connecting the endpoints of α, say x and y, and we claim that α can be transformed into ζ by flipping 4-cycles. We argue by induction on the length of α. If α shares an edge with ζ, there is nothing to prove. Otherwise, let $a \in \alpha$ and $b \in \zeta$ denote the two neighbours of x. Notice that the median point m of y, a, b yields the fourth vertex of a 4-cycle spanned by x, a, b.



As a consequence of our induction hypothesis, α can be transformed as a geodesic of the form $[x, a] \cup [a, m] \cup [m, y]$. By flipping a 4-cycle, we get the path $[x, b] \cup [b, m] \cup [m, y]$. By applying our induction hypothesis one more time, we can transform this path into ζ. $\square$

Median graphs for arbitrary varieties. We have seen in the second lecture that the blow-up graph does not generalize to varieties of dimension 3 or larger. However, we can adapt the idea of the construction by changing the setting in the following way:

- work with varieties that are not necessarily complete anymore,
- instead of blowing up points/contracting curves, we add/remove subvarieties.

Let X be a normal variety over a field k, i.e., an integral and separated scheme of finite type. We will construct now a median graph $\mathcal{C}^0(X)$ with an isometric action of $\mathrm{Bir}(X)$. We define the vertices of $\mathcal{C}^0(X)$ to be equivalence classes of marked pairs (A, φ), where A is a normal variety and $\varphi\colon A \dashrightarrow X$ is a birational map. Two marked pairs (A, φ) and (B, ψ) are equivalent if $\varphi^{-1}\psi\colon B \dashrightarrow A$ is an isomorphism in codimension 1.

Two vertices v and w are connected by an edge oriented from v to w if there exists a variety A with a marking $\varphi\colon A \dashrightarrow X$ and hypersurface $H \subset A$ such that v can be represented by (A, φ) and w by $(A \setminus H, \varphi|_{A\setminus H})$.

Example 50. If $\pi\colon Y \to X$ is a blow-up of a smooth variety of codimension ≥ 2, then (Y, π) and (X, id) are connected by an edge.

Remark 51. If S is a regular projective surface, then the blow-up graph is a subgraph of $\mathcal{C}^0(X)$. However, the embedding is not isometric.

Theorem 52. *The graph $\mathcal{C}^0(X)$ is a median graph.*

Idea of the proof of Theorem 52. Let us consider the cube completion $\mathcal{C}^0(X)^\square$ of $\mathcal{C}^0(X)$. There is an n-cube between 2^n distinct vertices $v_1, \dots, v_{2^n}$ in $\mathcal{C}^0(X)^\square$ if the vertices can be represented by $v_i = [(A_i, \varphi_i)]$ such that (up to reordering) the following is satisfied:

1. A_1 contains hypersurfaces $H_1, \dots, H_n$, such that for each subset $I \subset 1, \dots, n$ there is a j such that $A_j = A_1 \setminus \{\cup_{r \in I} H_r\}$,

2. the marking φ_j is just the restriction of φ_1 to A_j.

One can show that every finite set of vertices $\{v_1, \dots, v_n\}$ is contained in a cube in $\mathcal{C}^0(X)^\square$.

This implies directly that $\mathcal{C}^0(X)^\square$ is simply connected and that the links of all its vertices are flag. Hence, the graph $\mathcal{C}^0(X)$ is median. $\square$

We now describe the hyperplanes in $\mathcal{C}^0(X)$. An edge in $\mathcal{C}^0(X)$ is given by removing an irreducible hypersurface H of from a marked variety (A, φ). We denote the hyperplane defined by this edge by $[(A, \varphi, H)]$. One can check the following:

Lemma 53. *We have that* $[(A, \varphi, H)] = [(B, \psi, K)]$ *if and only if* K *is not contained in the exceptional locus of* $\varphi^{-1}\psi$ *and* H *is the strict transform* $\varphi^{-1}\psi(K)$ *of* K*.*

An isometric action of $\mathrm{Bir}(X)$**.** Similarly as for the blow-up complex, we obtain an isometric actions of $\mathrm{Bir}(X)$ on $\mathcal{C}^0(X)$. Namely, an element $f \in \mathrm{Bir}(X)$ maps a vertex $[A, \varphi]$ of $\mathcal{C}^0(X)$ to the vertex $[A, f\varphi]$. It is straightforward to check that this is well defined and defines an orientation preserving isometric action of $\mathrm{Bir}(X)$ on $\mathcal{C}^0(X)$.

Let Y be a variety and $f \in \mathrm{Bir}(Y)$. We denote the *exceptional locus* of f by $\mathrm{Exc}(f)$. Denote by $\mathrm{Exc}^1(f)$ the set of hypersurfaces of Y contained in $\mathrm{Exc}(f)$. An $f \in \mathrm{Bir}(X)$ is a *pseudo-automorphism* of Y if $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f^{-1})$ are of codimension ≥ 2, in other words, if $\mathrm{Exc}^1(f)$ and $\mathrm{Exc}^1(f^{-1})$ are empty. The group of pseudo-automorphisms of Y is denoted by $\mathrm{Psaut}(Y)$. Let us make the following observations:

- A subgroup $G \subset \mathrm{Bir}(X)$ fixes a vertex $[A, \varphi]$ if and only if $\varphi^{-1}G\varphi \subset \mathrm{Psaut}(A)$.
- We have $d([A, \varphi], [B, \psi]) = |\mathrm{Exc}^1(\psi^{-1}\varphi)| + |\mathrm{Exc}^1(\varphi^{-1}\psi)|$.

By applying again Haglund's classification of isometries and the fact that our isometric action preserves the orientation, we obtain that every element in $\mathrm{Bir}(X)$ either fixes a vertex or acts by translation along an axis. We therefore obtain:

Corollary 54. *Let* $f \in \mathrm{Bir}(X)$*. Then the number* $|\mathrm{Exc}^1(f^n)| + |\mathrm{Exc}^1(f^{-n})|$ *is uniformly bounded for all* n*, or it grows asymptotically linearly in* n*.*

Remark 55. Corollary 54 can be used to give new constraints on the degree growth of birational transformations as well as on the structure of centralizers of generic elements.

A subgroup $G \subset \mathrm{Bir}(X)$ is *pseudo-regularisable* if there exists a variety Y and a birational map $\varphi\colon X \dashrightarrow Y$ such that $\varphi^{-1}G\varphi \subset \mathrm{Psaut}(Y)$. The theorem of Gerasimov implies the following:

Corollary 56. *A subgroup* $G \subset \mathrm{Bir}(X)$ *is pseudo-regularisable if and only if the number* $|\mathrm{Exc}^1(f^n)| + |\mathrm{Exc}^1(f^{-n})|$ *is uniformly bounded for all* $f \in G$*.*

Constructing quotients We apply now the construction from the beginning of this lecture to construct quotients of $\mathrm{Bir}(X)$.

Consider the $\mathrm{Bir}(X)$ action on the hyperplanes in $\mathcal{C}^0(X)$. An element $f \in \mathrm{Bir}(X)$ maps a hyperplane $[(A, \varphi, H)]$ to the hyperplane $[(A, f\varphi, H)]$. In particular, a hyperplane of the form $[(X, \varphi, H)]$ is in the same $\mathrm{Bir}(X)$-orbit as a hyperplane of the form $[(X, \psi, K)]$ if and only if there exists a birational transformation $f \in \mathrm{Bir}(X)$ such that the strict transform $f(H) = K$, i.e., the two irreducible hypersurfaces H and K are *Cremona equivalent.*

Theorem 57. *Let X be a normal variety and let $f \in \mathrm{Bir}(X)$. Let $H_1, \ldots, H_k$ be the irreducible components of strict codimension 1 of the exceptional locus of f and let $K_1, \ldots, K_m$ be the irreducible components of strict codimension 1 of the exceptional locus of f^{-1}.*

If there exists an H_i that is not Cremona equivalent to any of the K_j, then there exists a homomorphism $\varphi\colon \mathrm{Bir}(X) \to \mathbb{Z}$ such that $f \notin \ker(\varphi)$. Moreover, $\mathrm{Bir}(X)$ is not generated by pseudo-regularisable elements.

Proof. Fix the vertex $o = [(X, \mathrm{id})]$ in $\mathcal{C}^0(X)$. The path from o to $f(o)$ crosses the hyperplanes $[(X, \mathrm{id}, H_k)]$ with positive sign and the hyperplanes $[(X, f^{-1}, K_l)]$ with negative sign. Since H_i is not Cremona equivalent to any of the K_j, this implies that the path from o to $f(o)$ passes the $\mathrm{Bir}(X)$-orbit of the hyperplane $[(X, \mathrm{id}, H_i)]$ only with positive signs. Hence $\Theta(f) \neq 0$.

Every pseudo-regularisable element fixes a vertex in $\mathcal{C}^0(X)$. Hence, all pseudo-regularisable elements are in the kernel of Θ. Therefore, they cannot generate $\mathrm{Bir}(X)$ in this case. □

In [LS22] the authors construct for many important classes of fields and for $n \geq 4$ elements $f \in \mathrm{Bir}(\mathbb{P}^n_k)$ that satisfy the conditions from Theorem 57. More precisely, they show that there exist $f \in \mathrm{Bir}(\mathbb{P}^n_k)$ whose exceptional locus contains an irreducibles hypersurface that is not birationally equivalent to any irreducible hypersurface contained in the exceptional locus of f^{-1}. This generalizes a result from [HL18]. The authors then use motivic arguments to construct a non-trivial homomorphism $c\colon \mathrm{Bir}(\mathbb{P}^n_k) \to \mathbb{Z}$.

Theorem 57 gives a perspective on the work of Shinder and Lin using the geometry of our median graphs instead of the motivic invariant c.

Corollary 58. *Let k be an infinite field and $n \geq 4$. There exists a non-trivial homomorphism $\varphi\colon \mathrm{Bir}(\mathbb{P}^n_k) \to \mathbb{Z}$. Moreover, $\mathrm{Bir}(\mathbb{P}^n_k)$ is not generated by pseudo-regularisable elements.*

We would like to emphasize that the main work in [LS22] and [HL18] consists in constructing the examples of $f \in \mathrm{Bir}(\mathbb{P}^n_k)$ satisfying the required properties, and not in constructing the homomorphism c. So the geometry of median graphs cannot substitute arguments from algebraic geometry in this instance.

However, the upshot of our point of view is that we do not require the existence of $f \in \mathrm{Bir}(X)$ whose exceptional locus contains an irreducibles hypersurface that is not *birationally equivalent* to any irreducible hypersurface contained in the exceptional locus of f^{-1}, but only the existence of $f \in \mathrm{Bir}(X)$ whose exceptional locus contains an irreducible hypersurface that is not *Cremona equivalent* to any irreducible hypersurface contained in the exceptional locus of f^{-1}. Cremona equivalence for hypersurfaces in $\mathbb{P}^n$ is a much weaker notion than birational equivalence (see [MP12]). Hence, this strategy could be used to construct non-trivial homomorphisms $\mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}}) \to \mathbb{Z}$. At this point, it is still an open question whether such homomorphisms exist.

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