ACTIONS OF CREMONA GROUPS ON CAT(0) CUBE COMPLEXES

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1. Introduction

To an algebraic variety X we associate its group of birational transformations $\mathrm{Bir}(X)$. The group structure of $\mathrm{Bir}(X)$ has been an active area of research since more than 150 years. In the last decades, geometric group theory and dynamics have opened up new viewpoints on the subject. If S is a surface, $\mathrm{Bir}(S)$ acts by isometries on an infinite dimensional hyperbolic space, which has been used, for instance, to show that $\mathrm{Bir}(S)$ satisfies the Tits alternative ([Can11, Ure17]), that $\mathrm{Bir}(\mathbb{P}^2_k)$ is not simple (see [CLC13, Lon16, Ure20]), or to study the degree growth of birational transformations of surfaces (see [Can15] for an overview). Recently, in geometric group theory, CAT(0) cube complexes turned out to be a useful tool to study large families of groups (see for instance [Gen22]). A cube complex is CAT(0) if it is simply connected and locally non-positively curved. More precisely, a cube complex is CAT(0) if it is simply connected and if the link of every vertex is flag. It turned out that CAT(0) cube complexes also provide new insights into groups of birational transformations of the plane as well as groups of birational transformations in arbitrary dimensions ([LU21, GLU23a, LPU23, GLU23b]).

In this text, we review the *blow-up complex* $\mathscr{C}(S)$, which is defined in [LU21] for regular projective surfaces S together with a non-exhaustive list of examples of old and new results, which can be deduced from these cube complexes. We will also briefly explain the idea of constructions of CAT(0) cube complexes with isometric actions of $\mathrm{Bir}(X)$ for varieties X of arbitrary dimension. All the constructions and results work over arbitrary base-fields k.

Remark 1.1. In a more modern terminology, it is common to rather speak of *median graphs*, which are exactly the 1-skeletons of CAT(0) cube complexes, instead of CAT(0) cube complexes. This is for various reasons more natural ([Gen23]). In order to avoid confusion, we will nevertheless use the vocabulary of CAT(0) cube complexes as it is used in [LU21]. In this text, we always work with the combinatorial metric on the vertices.

2. The blow-up complex

Let S be a projective regular surface over k. We consider *marked surfaces*, i.e., pairs (S',φ), where S' is a regular projective surface and $\varphi\colon S' \dashrightarrow S$ a birational map.

The *blow-up complex* $\mathscr{C}(S)$ is the following cube complex:

- The vertices of $\mathscr{C}(S)$ are given by equivalence classes of marked surfaces, where $(T,\varphi)\sim(T',\varphi')$ if the map $\varphi'^{-1}\varphi\colon T\to T'$ is an isomorphism,
- A set $\{v_1,\dots,v_{2n}\}$ of distinct vertices spans a n-cube if the v_j can be represented by (T_j,φ_j) and if there is $1\leq r\leq 2^n$ and n distinct closed points $p_1,\dots,p_n\in T_j$ such that for any $1\leq j\leq 2^n$ the map $\varphi_r^{-1}\varphi_j\colon T_j\to T_r$ is the blow-up of a subset of points of $\{p_1,\dots,p_n\}$.

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Let us note that $\mathscr{C}(S)$ is infinite dimensional. The proof of the following result is not hard, but central:

Theorem 2.1 ([LU21, Theorem 1.1])**.** *The cube complexes $\mathscr{C}(S)$ are CAT(0).*

The group $\mathrm{Bir}(S)$ acts on $\mathscr{C}(S)$ by isometries and without inversions. An element $f \in \mathrm{Bir}(S)$ maps a vertex represented by (T, φ) to the vertex represented by $(T, f\varphi)$.

An isometry f of a CAT(0) cube complex is called *elliptic* if it fixes a vertex and *loxodromic* if it acts by translation along a combinatorial geodesic, which is called the *axis* of f.

Proposition 2.2 ([Hag07])**.** *Every isometry f of a* CAT(0) *cube complex such that f and each of its iterates act without inversion is either loxodromic or elliptic.*

Another useful result about groups acting on CAT(0) cube complexes is the following:

Proposition 2.3 ([Ger98], [Rol98])**.** *If a group acts isometrically on a* CAT(0) *cube complex without inversions and with a bounded orbit, then G fixes a vertex.*

A group G is said to have *property FW* if every isometric action of G on a CAT(0) cube complex fixes a point (see [Cor13]). For instance, $\mathrm{SL}_n(\mathbb{Z})$ has property FW if $n \geq 3$. Property FW is weaker than Kazhdan's property (T).

One observes the following facts:

- The distance $d(v_1, v_2)$ between two vertices $v_1 = [T_1, \varphi_1]$ and $v_2 = [T_2, \varphi_2]$ is exactly $\#\,\mathrm{bp}(\varphi_1^{-1}\varphi_2) + \#\,\mathrm{bp}(\varphi_2^{-1}\varphi_1)$, where $\#\,\mathrm{bp}(f)$ denotes the number of base-points of a birational map between regular projective surfaces.
- A subgroup $G \subset \mathrm{Bir}(S)$ fixes a vertex (T, φ) if and only if $\varphi^{-1} f \varphi \in \mathrm{Aut}(T)$ for all $f \in G$. In other words, elliptic elements in $\mathrm{Bir}(S)$ correspond exactly to *projectively regularizable* transformations, i.e., transformations that are conjugate to automorphisms of regular projective surfaces.
- If a vertex $v = [T, \varphi]$ lies on the axis of a loxodromic element $f \in \mathrm{Bir}(S)$ then $d(v, f^n(v)) = 2|n| d(v, f(v)) = 2|n| \#\,\mathrm{bp}(\varphi^{-1} f \varphi)$ for all $n \in \mathbb{Z}$. This implies in particular that f is *algebraically stable* (see [DF01]).

These observations together with Theorem 2.1 and the Propositions 2.2 and 2.3 imply the following theorem, which generalizes and reproves results from [DF01], [BD15], and [CC19].

Theorem 2.4. *Let S be a regular projective surface and $f \in \mathrm{Bir}(S)$. Then the following is true:*

1. *There exists an algebraically stable model for f,*
2. *$\#\,\mathrm{bp}(f^n) \sim cn$ for some $c \in \mathbb{Z}_{\geq 0}$,*
3. *a subgroup $G \subset \mathrm{Bir}(S)$ is projectively regularizable if and only if $\#\,\mathrm{bp}(f)$ is uniformly bounded for all $f \in G$,*
4. *if $G \subset \mathrm{Bir}(S)$ is a subgroup with property FW, then G is projectively regularizable,*
5. *every divisible and distorted element in* $\mathrm{Bir}(S)$ *is projectively regularizable.*

3. On generalizations to higher dimensions

The blow-up complex unfortunately does not allow direct generalizations to higher dimensions. However, we can construct a CAT(0) cube complex $\mathcal{C}^0(X)$ for arbitrary normal varieties X with an isometric action of $\mathrm{Bir}(X)$. In this case, however, the vertices are classes of marked varieties that are not necessarily complete, as it was the case for the blow-up complex and hence the vertex stabilizers are much larger. Nevertheless, we can apply similar arguments as for the blow-up complex. This time, we obtain results about the dynamical behavior of $\#\operatorname{Exc}(f^n)$ for $f \in \operatorname{Bir}(X)$, where $\#\operatorname{Exc}(g)$ denotes the number of irreducible components of codimension one of the exceptional locus of a birational map g of X. This gives us, for instance, new constraints on the degree growth of certain transformations, or it can be used to show that the centralizer of many elements in $\operatorname{Bir}(X)$ is relatively small.

This construction generalizes and yields a series of CAT(0) cube complexes $\mathcal{C}^\ell$ for $0 \leq \ell < \dim(X)$, which allow us to show for instance that subgroups of $\operatorname{Bir}(X)$ with property FW are always conjugate to subgroups of $\operatorname{Aut}(Y)$ for some, not necessarily complete, variety Y, thus generalizing results from [CX18, Cor20].

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