Subgroups of elliptic elements of the plane Cremona group

Christian Urech

The main source for the content of this abstract are the two papers [7] and [8]. The *Cremona group* $\mathrm{Cr}_2(\mathbb{C})$ is the group of birational transformation of the complex projective plane. One of the key techniques for studying the group theoretical properties of infinite subgroups of the complex plane Cremona group $\mathrm{Cr}_2(\mathbb{C})$ has been an action by isometries on an infinite dimensional hyperboloid $\mathbb{H}^\infty(\mathbb{P}^2)$ (see [3] for an overview and references). Recall that there are three types of isometries of hyperbolic spaces:

- *elliptic isometries*, which are the isometries that fix a point in $\mathbb{H}^\infty(\mathbb{P}^2)$,
- *parabolic isometries*, which are the isometries that do not fix any point in $\mathbb{H}^\infty(\mathbb{P}^2)$, but fix exactly one point in the boundary $\partial\,\mathbb{H}^\infty(\mathbb{P}^2)$,
- *loxodromic isometries*, which are the isometries that do not fix any point in $\mathbb{H}^\infty(\mathbb{P}^2)$, but fix exactly two points in $\partial\,\mathbb{H}^\infty(\mathbb{P}^2)$.

We call an element $f \in \mathrm{Cr}_2(\mathbb{C})$ *elliptic, parabolic* or *loxodromic*, if the isometry of $\mathbb{H}^\infty(\mathbb{P}^2)$ induced by f is elliptic, parabolic or loxodromic respectively. This notion is linked to the dynamical behavior of f.

We consider subgroups of $\mathrm{Cr}_2(\mathbb{C})$ consisting only of elliptic elements. The main result is that the group theoretical structure of these subgroups is not more complicated than the structure of algebraic subgroups of $\mathrm{Cr}_2(\mathbb{C})$:

Theorem 0.1 ([8])**.** *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup of elliptic elements. Then one of the following is true:*

(1) *G is contained in an algebraic subgroup;*
(2) *G preserves a rational fibration;*
(3) *G is a torsion subgroup.*

Theorem 0.2 ([8])**.** *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a torsion subgroup. Then G is isomorphic to a bounded subgroup of* $\mathrm{Cr}_2(\mathbb{C})$.

In combination with the classification of maximal algebraic subgroups (see [1]), Theorem 0.1 and Theorem 0.2 give an explicit description of groups of elliptic elements. This allows to give new descriptions of arbitrary subgroups of $\mathrm{Cr}_2(\mathbb{C})$.

Theorem 0.1 and Theorem 0.2 can now be used to prove structure theorems on general subgroups of $\mathrm{Cr}_2(\mathbb{C})$. Given a subgroup G of $\mathrm{Cr}_2(\mathbb{C})$ one can consider the following three cases:

(1) G contains a loxodromic element;
(2) G contains no loxodromic element but a parabolic element;
(3) G is a subgroup of elliptic elements.

In case (1), the group G can be understood by using tools from hyperbolic geometry and geometric group theory, in case (2) it is known that G preserves a rational or elliptic fibration and case (3) can be treated with the help of Theorem 0.1 and Theorem 0.2. Let us explain two results that can be proved with this strategy.

1. The Tits alternative

Recall the following definition:

Definition 1.1.

(1) *A group G satisfies the Tits alternative if every subgroup of G is either virtually solvable or contains a non-abelian free subgroup.*

(2) *A group G satisfies the Tits alternative for finitely generated subgroups if every finitely generated subgroup of G either is virtually solvable or contains a non-abelian free subgroup.*

Cantat established the Tits alternative for finitely generated subgroups of $\mathrm{Cr}_2(\mathbb{C})$ ([2]). Theorem 0.1 and Theorem 0.2 yield the results needed to generalize this result:

Theorem 1.1 ([8]). *The plane Cremona group $\mathrm{Cr}_2(\mathbb{C})$ satisfies the Tits alternative.*

2. Simple subgroups of the plane Cremona group

It had been a long-standing open question, whether the plane Cremona group is simple as a group until Cantat and Lamy showed in 2012 that it is not ([4]). The main idea to prove this result was to use techniques from small cancellation theory, an approach that has been refined by Shepherd-Barron and Lonjou (see [6], [5]). These results are a starting point for the following classification of all simple subgroups of the plane Cremona group:

Theorem 2.1 ([7]). *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a simple group. Then:*

(1) *G does not contain loxodromic elements.*

(2) *If G contains a parabolic element, then G is conjugate to a subgroup of $\mathcal{J}$.*

(3) *If all elements in G are elliptic, then either G is a simple subgroup of an algebraic subgroup of $\mathrm{Cr}_2(\mathbb{C})$, or G is conjugate to a subgroup of $\mathcal{J}$.*

With the help of Theorem 2.1 all simple groups that act non-trivially by birational transformations on compact complex Kähler surfaces can be described:

Theorem 2.2 ([7]). *Let G be a simple group. Then*

(1) *G acts non-trivially by birational transformations on a rational complex projective surface if and only if G is isomorphic to a subgroup of $\mathrm{PGL}_3(\mathbb{C})$.*

(2) *G acts non-trivially by birational transformations on a non-rational compact complex Kähler surface of negative Kodaira dimension if and only if G is finite or isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C})$.*

(3) *G acts non-trivially by birational transformations on a compact complex Kähler surface S of non-negative Kodaira dimension if and only if G is finite.*

References

[1] J. Blanc, *Sous-groupes algébriques du groupe de Cremona*, Transform. Groups **14** (2009), no. 2, 249–285.

[2] S. Cantat, *Sur les groupes de transformations birationnelles des surfaces*, Ann. of Math. (2) **174** (2011), no. 1, 299–340.

[3] S. Cantat, *The Cremona groups*, to appear in Proceedings of 2015 Summer Institute on Algebraic Geometry, AMS Proceedings of Symposia in Pure Mathematics (2015).

[4] S. Cantat, S. Lamy, and Y. de Cornulier, *Normal subgroups in the cremona group*, Acta mathematica **210** (2013), no. 1, 31–94.

[5] A. Lonjou, *Non simplicité du groupe de Cremona sur tour corps*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 5, 2021–2046. MR 3533276

[6] N.I. Shepherd-Barron, *Some effectivity questions for plane cremona transformations*,arXiv preprint, arXiv:1311.6608 (2013).

[7] C. Urech, *Simple groups of birational transformations in dimension two*, arXiv preprint, arXiv:1802.09258 (2018).

[8] C. Urech, *Subgroups of elliptic elements of the cremona group*, arXiv preprint, arXiv:1802.08485 (2018).